

# Set Theory Revisited

# Outline for Today

- ***Proofs on Sets***
  - Making our intuitions rigorous.
- ***Formal Set Definitions***
  - What do our terms mean?
- ***Appendices: Examples***
  - Sample proofs to help you get the hang of the ideas here.

Recap from Last Time

	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$\forall x. A$	Initially, <i>do nothing</i> . Once you find a $z$ through other means, you can state it has property $A$ .	Have the reader pick an arbitrary $x$ . We then prove $A$ is true for that choice of $x$ .
$\exists x. A$	Introduce a variable $x$ into your proof that has property $A$ .	Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$ .
$A \rightarrow B$	Initially, <i>do nothing</i> . Once you know $A$ is true, you can conclude $B$ is also true.	Assume $A$ is true, then prove $B$ is true.
$A \wedge B$	Assume $A$ . Also assume $B$ .	Prove $A$ . Also prove $B$ .
$A \vee B$	Consider two cases. Case 1: $A$ is true. Case 2: $B$ is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$ . <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$ .	Prove $A \rightarrow B$ and $B \rightarrow A$ .
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

New Stuff!

# Proving Results from Set Theory

**Claim:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \in B$  and  $B \in C$ , then  $A \in C$ .

**Proof (?):** Assume  $A$ ,  $B$ , and  $C$  are sets where  $A \in B$  and  $B \in C$ .

We need to show that  $A \in C$ .

Since  $A \in B$ , we know that  $A$  is contained in  $B$ . Since  $B \in C$ , we know that  $B$  is contained in  $C$ . Therefore, because  $A$  is contained in  $B$  and  $B$  is contained in  $C$ , we know that  $A$  is contained in  $C$ . This means that  $A \in C$ , as required. ■

**Claim:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Proof (?):** Assume  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ .

We need to show that  $A \subseteq C$ .

Since  $A \subseteq B$ , we know that  $A$  is contained in  $B$ . Since  $B \subseteq C$ , we know that  $B$  is contained in  $C$ . Therefore, because  $A$  is contained in  $B$  and  $B$  is contained in  $C$ , we know that  $A$  is contained in  $C$ . This means that  $A \subseteq C$ , as required. ■

Which (if any) of these claims are true?  
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**Lie:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \in B$  and  $B \in C$ , then  $A \in C$ .

**Bad Proof:** Assume  $A$ ,  $B$ , and  $C$  are sets where  $A \in B$  and  $B \in C$ .  
We need to show that  $A \in C$ .

Since  $A \in B$ , we know that  $A$  is contained in  $B$ . Since  $B \in C$ , we know that  $B$  is contained in  $C$ . Therefore, because  $A$  is contained in  $B$  and  $B$  is contained in  $C$ , we know that  $A$  is contained in  $C$ . This means that  $A \in C$ , as required. ■

**Theorem:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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This can't be a good proof;  
the same basic argument  
proves a false claim!

**Claim:** Let  $A$ ,  $B$ , and  $C$  be sets. If  $A \subseteq B$  and  $A \subseteq C$ , then we have  $A \subseteq B \cap C$ .

**Proof (?):** Assume  $A \subseteq B$  and  $A \subseteq C$ . We need to show  $A \subseteq B \cap C$ .

Since  $A \subseteq B$ , all elements of  $A$  are in  $B$ . Since  $A \subseteq C$ , all elements of  $A$  are also in  $C$ . Therefore, all elements of  $A$  are in both  $B$  and  $C$ . Therefore, we see that  $A \subseteq B \cap C$ . ■

**Claim:** Let  $A$ ,  $B$ , and  $C$  be sets. If  $A \subsetneq B$  and  $A \subsetneq C$ , then we have  $A \subsetneq B \cap C$ .

**Proof (?):** Assume  $A \subsetneq B$  and  $A \subsetneq C$ . We need to show  $A \subsetneq B \cap C$ .

Since  $A \subsetneq B$ , all elements of  $A$  are in  $B$  and there are other elements of  $B$ . Since  $A \subsetneq C$ , all elements of  $A$  are also in  $C$  and there are other elements of  $C$ . Therefore, all elements of  $A$  are in both  $B$  and  $C$ , and there are some other elements in  $B$  and  $C$ . Therefore, we see that  $A \subsetneq B \cap C$ . ■

(Reminder:  $S \subsetneq T$  means  
 $S \subseteq T$  and  $S \neq T$ .)

Which (if any) of these claims are true?  
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**Theorem:** Let  $A$ ,  $B$ , and  $C$  be sets. If  $A \subseteq B$  and  $A \subseteq C$ , then we have  $A \subseteq B \cap C$ .

**Bad Proof:** Assume  $A \subseteq B$  and  $A \subseteq C$ . We need to show  $A \subseteq B \cap C$ .

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(Reminder:  $S \subsetneq T$  means  
 $S \subseteq T$  and  $S \neq T$ .)

# What Went Wrong?

- The style of arguments you've just seen are **not** how to prove results on sets.
- As you've seen:
  - The reliance on high-level terms like “contained” is not mathematically precise.
  - A discussion of “all elements” of a set is not how to reason about collections of objects.
- **Question:** How do we write rigorous proofs about sets?

# The Importance of Definitions

- As you've seen this week, formal definitions underpin mathematical proofs.
- The major issue from the previous proofs is that we haven't defined what our terms mean.
  - How do we define what  $A \in B$  means?
  - How do we define what  $A \subseteq B$  means?
  - How do we define what  $A \cap B$  means?
- Think back to our proof triangle: we currently have intuitions for these concepts, but not formal definitions.

	Is defined as...	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$S \subseteq T$			
$S = T$			
$x \in S \cap T$			
$x \in S \cup T$			
$X \in \wp(S)$			
$x \in \{y \mid P(y)\}$			

# Proofs on Subsets

***Theorem:*** If  $A$ ,  $B$ , and  $C$  are sets where  
 $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

# Defining Subsets

- Formally speaking, if  $S$  and  $T$  are sets, we say that  $S \subseteq T$  when the following holds:

$$\forall x \in S. x \in T$$

- Now, suppose you're working with a proof where you encounter  $S \subseteq T$ . Think back to the proof table.
  - To **assume** that  $S \subseteq T$ , what should you do?
  - To **prove** that  $S \subseteq T$ , what should you do?

Answer at

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# A Correct Proof on Sets

**Theorem:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Proof:** Let  $A$ ,  $B$ , and  $C$  be sets where  $A \subseteq B$  and  $B \subseteq C$ . We need to prove that  $A \subseteq C$ . To do so, pick some  $x \in A$ ; we need to show that  $x \in C$ .

Since  $x \in A$  and  $A \subseteq B$ , we know  $x \in B$ . Similarly, since  $x \in B$  and  $B \subseteq C$ , we see that  $x \in C$ , as required. ■

**Theorem:** If  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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**Bad Proof:** Assume  $A$ ,  $B$ , and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ . We need to show that  $A \subseteq C$ .

Since  $A \subseteq B$ , we know that  $A$  is contained in  $B$ . Since  $B \subseteq C$ , we know that  $B$  is contained in  $C$ . Therefore, because  $A$  is contained in  $B$  and  $B$  is contained in  $C$ , we know that  $A$  is contained in  $C$ . This means that  $A \subseteq C$ , as required. ■

# Unions and Intersections

***Theorem:*** Let  $A$ ,  $B$ , and  $C$  be sets. If  $A \subseteq B$  and  $A \subseteq C$ , then  $A \subseteq B \cap C$ .

# Unions and Intersections

- The statement  $x \in S \cap T$  is defined as

$$x \in S \quad \wedge \quad x \in T.$$

- The statement  $x \in S \cup T$  is defined as

$$x \in S \quad \vee \quad x \in T.$$

- These are operational definitions: they show how unions and intersections interact with the  $\in$  relation rather than saying what the union or intersection of two sets “are.”

**Theorem:** Let  $A$ ,  $B$ , and  $C$  be sets. If  $A \subseteq B$  and  $A \subseteq C$ , then  $A \subseteq B \cap C$ .

**Proof:** Assume  $A \subseteq B$  and  $A \subseteq C$ . We need to prove that  $A \subseteq B \cap C$ . So pick some  $x \in A$ ; we need to show that  $x \in B \cap C$ .

Since  $x \in A$  and  $A \subseteq B$ , we know  $x \in B$ . Since  $x \in A$  and  $A \subseteq C$ , we know  $x \in C$ . Therefore, we see that  $x \in B$  and  $x \in C$ , so  $x \in B \cap C$ , as required. ■

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**Bad Proof:** Assume  $A \subseteq B$  and  $A \subseteq C$ . We need to show  $A \subseteq B \cap C$ .

Since  $A \subseteq B$ , all elements of  $A$  are in  $B$ . Since  $A \subseteq C$ , all elements of  $A$  are also in  $C$ . Therefore, all elements of  $A$  are in both  $B$  and  $C$ . Therefore, we see that  $A \subseteq B \cap C$ . ■

# Set Equality

***Theorem:*** Let  $A$  and  $B$  be sets. Then  $A \subseteq B$  if and only if  $A \cup B = B$ .

***Proof:*** See appendix!

**Time-Out for Announcements!**

# Problem Set Three

- Problem Set Two was due today at 1:00PM.
  - Need more time? You can use a late day to extend the deadline to Saturday at 1:00PM.
- Problem Set Three goes out today and is due next Friday at 1:00PM.
  - ***Start early!*** This gives you more time to make slow and steady progress.
  - Use the techniques from this week: the assume/prove table, the two-column organizer, etc.

Did you know that 20% of TAs  
are lonely in their office hours?

Stop by Sunday or Monday OH! Get your  
questions answered, meet cool people,  
have a good time.

Back to CS103!

# Set-Builder Notation

# Set-Builder Notation

- Let  $S$  be the set defined here:

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \geq 137 \}$$

- Now imagine you have some quantity  $x$ . Based on this...
  - ... if you **assume** that  $x \in S$ , what does that tell you about  $x$ ?
  - ... if you need to **prove** that  $x \in S$ , what do you need to prove?

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# Set-Builder Notation

- Like unions and intersections, we have an operational definition for set-builder notation. It's the following:

**Let  $S = \{ y \mid P(y) \}$ .**  
**Then  $x \in S$  when  $P(x)$  is true.**

- So, for example:
  - $x \in \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$  means  $x \in \mathbb{N}$  and  $x$  is even.
  - $x \in \{ n \mid \exists k \in \mathbb{N}. n = 2k + 1 \}$  means that there is a  $k \in \mathbb{N}$  where  $x = 2k + 1$ . (Equivalently,  $x$  is an odd natural number)
- **Key Point:** The placeholder variable disappears in all these examples. After all, *it's just a placeholder.*

# Proofs on Set-Builder Notation

# Some Useful Notation

- If  $n$  is a natural number, we define the set  **$[n]$**  as follows:

$$[n] = \{ k \mid k \in \mathbb{N} \wedge k < n \}$$

- So, for example:
  - $[3] = \{0, 1, 2\}$
  - $[0] = \emptyset$
  - $[5] = \{0, 1, 2, 3, 4\}$

***Theorem:*** If  $m, n \in \mathbb{N}$  and  $m < n$ ,  
then  $[m] \subseteq [n]$

**Theorem:** If  $m, n \in \mathbb{N}$  and  $m < n$ , then  $[m] \subseteq [n]$ .

**Proof:** Assume  $m$  and  $n$  are natural numbers where  $m < n$ . We need to show that  $[m] \subseteq [n]$ . To do so, pick some  $x \in [m]$ . We'll prove that  $x \in [n]$ .

Since  $x \in [m]$ , we know that  $x \in \mathbb{N}$  and  $x < m$ . Then, because  $x < m$  and  $m < n$ , we know that  $x < n$ .

Collectively this means that  $x \in \mathbb{N}$  and  $x < n$ , so  $x \in [n]$ , as required. ■

Notice that *there is no set-builder notation in this proof*. We were able to avoid it by using the rules for what  $x \in \{y \mid P(y)\}$  say to do.

	Is defined as...	If you <i>assume</i> this is true...	To <i>prove</i> that this is true...
$S \subseteq T$	$\forall x \in S. x \in T$	Initially, <i>do nothing</i> . Once you find some $z \in S$ , conclude $z \in T$ .	Ask the reader to pick an $x \in S$ . Then prove $x \in T$ .
$S = T$	$S \subseteq T \wedge T \subseteq S$	Assume $S \subseteq T$ and $T \subseteq S$ .	Prove $S \subseteq T$ . Also prove $T \subseteq S$ .
$x \in S \cap T$	$x \in S \wedge x \in T$	Assume $x \in S$ . Then assume $x \in T$ .	Prove $x \in S$ . Also prove $x \in T$ .
$x \in S \cup T$	$x \in S \vee x \in T$	Consider two cases: Case 1: $x \in S$ . Case 2: $x \in T$ .	Either prove $x \in S$ or prove $x \in T$ .
$X \in \wp(S)$	$X \subseteq S$ .	Assume $X \subseteq S$ .	Prove $X \subseteq S$ .
$x \in \{y \mid P(y)\}$	$P(x)$	Assume $P(x)$ .	Prove $P(x)$ .

# Your Action Items

- ***Read “Guide to Proofs on Discrete Structures.”***
  - There’s additional guidance and practice on the assume/prove dichotomy and how it manifests in problem-solving.
- ***Read “Discrete Structures Proofwriting Checklist.”***
  - Keep the items here in mind when writing proofs. We’ll use this when grading your problem set.
- ***Read “Guide to Proofs on Sets.”***
  - There’s some good worked examples in there to supplement today’s lecture, several of which will be relevant for the problem set.
- ***Start Problem Set 3.***
  - Start early and make slow and steady progress.

# Next Time

- ***Graph Theory***
  - A ubiquitous, powerful abstraction with applications throughout computer science.
- ***Vertex Covers***
  - Making sure tourists don't get lost.
- ***Independent Sets***
  - Helping the recovery of the California Condor.

## ***Appendix:*** More Sample Set Proofs

**Theorem:** Let  $A$ ,  $B$ ,  $C$ , and  $D$  be sets where  $A \subseteq C$  and  $B \subseteq D$ . Then  $A \cup B \subseteq C \cup D$ .

**Proof:** Pick some  $x \in A \cup B$ ; we need to show that  $x \in C \cup D$ .

Because  $x \in A \cup B$ , we know that  $x \in A$  or  $x \in B$ . We consider each case separately.

*Case 1:*  $x \in A$ . Since  $A \subseteq C$  and  $x \in A$ , we see that  $x \in C$ , and therefore that  $x \in C \cup D$ .

*Case 2:*  $x \in B$ . Then because  $B \subseteq D$  and  $x \in B$  we have  $x \in D$ , so  $x \in C \cup D$ .

In either case, we see that  $x \in C \cup D$ , as required. ■

**Theorem:** Let  $A$  and  $B$  be sets. Then  $A \subseteq B$  if and only if  $A \cup B = B$ .

**Proof:** We will prove each direction of implication.

( $\Rightarrow$ ) Assume  $A \subseteq B$ . We need to show that  $A \cup B = B$ . To do so, we need to show that  $A \cup B \subseteq B$  and that  $B \subseteq A \cup B$ .

First, we'll show  $A \cup B \subseteq B$ . Pick an  $x \in A \cup B$ . We need to show that  $x \in B$ . Since  $x \in A \cup B$ , we consider two cases:

Case 1:  $x \in A$ . Then since  $x \in A$  and  $A \subseteq B$ , we have  $x \in B$ .

Case 2:  $x \in B$ . Then by assumption  $x \in B$ .

Either way, we have  $x \in B$ , which is what we needed to show.

Next, we'll prove  $B \subseteq A \cup B$ . Pick some  $x \in B$ . Since  $x \in B$ , we know that  $x \in A \cup B$ , as required.

( $\Leftarrow$ ) Assume  $A \cup B = B$ . We need to show that  $A \subseteq B$ . So pick an  $x \in A$ ; we need to show that  $x \in B$ .

Since  $x \in A$ , we know that  $x \in A \cup B$ . And since  $x \in A \cup B$  and  $A \cup B = B$ , we see that  $x \in B$ , as required. ■

**Theorem:** Let  $A$  and  $B$  be sets. If  $\wp(A) = \wp(B)$ , then  $A = B$ .

**Proof:** Assume  $\wp(A) = \wp(B)$ . We need to show that  $A = B$ , or, equivalently, that  $A \subseteq B$  and  $B \subseteq A$ . Since the roles of  $A$  and  $B$  are symmetric, we'll just prove  $A \subseteq B$ .

Pick some  $x \in A$ ; we need to show that  $x \in B$ . Since  $x \in A$ , we know that  $\{x\} \subseteq A$ . This means that  $\{x\} \in \wp(A)$ , and since  $\wp(A) \subseteq \wp(B)$  we know  $\{x\} \in \wp(B)$ . Thus we see that  $\{x\} \subseteq B$ , which in turn means that  $x \in B$ , as required. ■